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Remarks on the multi-species exclusion process with reflective boundaries

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Abstract

We investigate one of the simplest multi-species generalizations of the one dimensional exclusion process with reflective boundaries. The Markov matrix governing the dynamics of the system splits into blocks (sectors) specified by the number of particles of each kind. We find matrices connecting the blocks in a matrix product form. The procedure (generalized matrix ansatz) to verify that a matrix intertwines blocks of the Markov matrix was introduced in the periodic boundary condition, which starts with a local relation (Arita et al, *J. Phys. A* **44**, 335004 (2011)). The solution to this relation for the reflective boundary condition is much simpler than that for the periodic boundary condition.

1 Introduction

The asymmetric simple exclusion process (ASEP) is a lattice-gas model of interacting particles [16], where each particle is a random walker hopping from a site to one of the neighboring locations only if the target site is empty. The ASEP in one dimensional lattice ($\mathbb{Z}$ or its subset) has been intensively studied as an exact solvable non-equilibrium model [7, 10, 25] which is related to growth phenomena [13, 14, 21] and applied to modeling of various transport systems "from molecules to vehicles" [17, 22]. One of standard generalizations of the ASEP to multi-species systems (N-species ASEP) where each site i takes a local state $k_i \in \{1, \dots, N+1\}$ ($N \geq 0$) is as follows; nearest neighbor pairs of local states $(\alpha, \beta) = (k_i, k_{i+1})$ are interchanged

$$\alpha\beta \to \beta\alpha \quad \text{with rate} \quad \begin{cases} 1 & (\alpha < \beta), \\ q & (\alpha > \beta), \end{cases} \tag{1}$$

where we impose $0 \leq q \leq 1$ without loss of generality. We say that the site i is occupied by an *αth-class particle* for $k_i = \alpha \leq N$. We also say that the site i is empty for $k_i = N+1$, which is usually denoted by 0. The usual ASEP corresponds to $N = 1$. We will be formally concerned with the zero-species ASEP ($N = 0$) as well. We will call the cases $q = 1$ and $q = 0$ multi-species symmetric simple exclusion process (SSEP) and multi-species totally ASEP (TASEP), respectively.

In [6], the spectral structures of Markov matrices (which govern the dynamics of the system) were clarified. On the other hand, in [19], the matrix product form for the stationary state was found in the periodic-boundary case. Then these two studies were combined in [4] where matrices connecting dynamics of different values of N were constructed. This generalizes the procedure of the matrix (product) ansatz for stationary states [7, 11, 12]. In particular this generalized matrix ansatz enables us to *transfer* information to a system consisting of N-species particles from simpler systems consisting of $N'(< N)$-species particles. The question is whether this generalized matrix ansatz is applicable to the *reflective-boundary* case or not. This will be answered positively, which is the main purpose of this paper.

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Here we define the model on the L-site chain precisely. Let $\{|1\rangle, \ldots, |N+1\rangle\}$ be the basis of the single-site space $\mathbb{C}^{N+1}$, and represent a particle configuration $k_1 \cdots k_L$ as the ket vector $|k_1 \cdots k_L\rangle = |k_1\rangle \otimes \cdots \otimes |k_L\rangle \in (\mathbb{C}^{N+1})^{\otimes L}$. We also use the corresponding bra vectors $\langle k|$ and $\langle k_1 \cdots k_L| = \langle k_1| \otimes \cdots \otimes \langle k_L|$ with $\langle j|k\rangle = \delta_{jk}$. In terms of the probability vector

$$|P(t)\rangle = \sum_{1 \leq k_i \leq N+1} P(k_1 \cdots k_L; t)|k_1 \cdots k_L\rangle, \tag{2}$$

with each coefficient $P(k_1 \cdots k_L; t)$ representing the probability of finding the configuration $k_1 \cdots k_L$ at time t, our model is governed by the master equation $\frac{d}{dt}|P(t)\rangle = M^{(N)}|P(t)\rangle$. The matrix $M^{(N)}$ has the form

$$M^{(N)} = \sum_{1 \leq i \leq L-1} \left(M^{(N)}_{\mathrm{Loc}}\right)_{i,i+1}, \tag{3}$$

$$M^{(N)}_{\mathrm{Loc}} = \sum_{\alpha,\beta=1}^{N+1} \left(-\Theta(\alpha-\beta)|\alpha\beta\rangle\langle\alpha\beta| + \Theta(\alpha-\beta)|\beta\alpha\rangle\langle\alpha\beta|\,\right), \tag{4}$$

where $\left(M^{(N)}_{\mathrm{Loc}}\right)_{i,i+1}$ acts nontrivially on the ith and $(i+1)$st components of the tensor product, and Θ corresponds to the transition rate

$$\Theta(\alpha-\beta) = \begin{cases} 1 & (\alpha < \beta), \\ q & (\alpha > \beta), \\ 0 & (\alpha = \beta). \end{cases} \tag{5}$$

We call the $(N+1)^L \times (N+1)^L$ matrix $M^{(N)}$ total Markov matrix or simply Markov matrix, and the $(N+1)^2 \times (N+1)^2$ matrix $M^{(N)}_{\mathrm{Loc}}$ local Markov matrix. The relevant two-dimensional vertex model is a special case of Perk-Schultz model [2, 3, 18, 23].

We emphasize that we will investigate the multi-species ASEP on the L-site closed segment with the *reflective boundary condition* (*free boundary condition*). I.e., we do not impose any boundary term in the Markov matrix (3). The total Markov matrix obviously preserves the number of particles of each class and thus it has the following block diagonal structure:

$$M^{(N)} = \bigoplus_m M_m, \quad M_m \in \mathrm{End}V_m, \quad (\mathbb{C}^{N+1})^{\otimes L} = \bigoplus_m V_m. \tag{6}$$

Here each sector V_m is specified by the number of particles of each class

$$V_m = \bigoplus_{\#\{i|k_i=j\}=m_j} \mathbb{C}|k_1 \cdots k_L\rangle, \tag{7}$$

for $m = (m_1, \ldots, m_{N+1})$. This means that each sector is spanned by the ket vectors corresponding to permutations of the sequence $\underbrace{1 \cdots 1}_{m_1}\underbrace{2 \cdots 2}_{m_2} \cdots \underbrace{N+1 \cdots N+1}_{m_{N+1}}$. We also call the label m for the sector V_m "sector". We write the dual space for each sector as

$$V_m^* = \bigoplus_{\#\{i|k_i=j\}=m_j} \mathbb{C}\langle k_1 \cdots k_L|. \tag{8}$$

This paper is organized as follows. In section 2 we review the symmetry of the model, and construct matrices which connect dynamics of different sectors. We call these connecting matrices "conjugation matrix", and in section 3 we will reconstruct them in a matrix product form. In section 4 we consider a relation between Markov matrices of different values of N, by introducing similar conjugation matrices. In section 5 we investigate the stationary state and the relaxation to it. We will see that the stationary state can be regarded as a product of conjugation matrices. Section 6 is the summary of this paper.

2 Symmetry

The Markov matrix is mapped to a $U_q(SU(N+1))$ invariant quantum Hamiltonian H by the similarity transformation [2, 3, 8, 9, 20, 24]

$$-\frac{1}{\sqrt{q}}SM^{(N)}S^{-1}=H^{(N)},\ S=\sum_{1\le k_i\le N+1}q^{\frac{1}{4}\sum_{1\le i<j\le L}\mathrm{sign}(k_i-k_j)}|k_1\cdots k_L\rangle\langle k_1\cdots k_L|, \tag{9}$$

$$H^{(N)}=\sum_{1\le i\le L-1}\left(H_{\mathrm{Loc}}^{(N)}\right)_{i,i+1},\ H_{\mathrm{Loc}}^{(N)}=-\sum_{\substack{1\le\alpha,\beta\le N+1\\ \alpha\neq\beta}}|\alpha\beta\rangle\langle\beta\alpha|+\sum_{1\le\alpha,\beta\le N+1}\frac{\Theta(\alpha-\beta)}{\sqrt{q}}|\alpha\beta\rangle\langle\alpha\beta|. \tag{10}$$

Thus $M^{(N)}$ also has a symmetry, which we review in this section. We define $U_q(SU(N+1))$ generators

$$f^{(n)}=|n+1\rangle\langle n|,\quad e^{(n)}=|n\rangle\langle n+1|,\quad k^{(n)}=\sqrt{q}|n\rangle\langle n|+\frac{1}{\sqrt{q}}|n+1\rangle\langle n+1|+\sum_{\substack{1\le x\le N+1\\ x\neq n,n+1}}|x\rangle\langle x| \tag{11}$$

and the comultiplications

$$\Delta f^{(n)}=f^{(n)}\otimes Id+k^{(n)}\otimes f^{(n)},\quad \Delta e^{(n)}=e^{(n)}\otimes(k^{(n)})^{-1}+Id\otimes e^{(n)},\quad \Delta k^{(n)}=k^{(n)}\otimes k^{(n)}, \tag{12}$$

which commutes with the local Hamiltonian:

$$[\Delta f^{(n)},H_{\mathrm{Loc}}^{(N)}]=[\Delta e^{(n)},H_{\mathrm{Loc}}^{(N)}]=[\Delta k^{(n)},H_{\mathrm{Loc}}^{(N)}]=0. \tag{13}$$

From these local relations, we find global commutation relations

$$[F^{(n)},H^{(N)}]=[E^{(n)},H^{(N)}]=[K^{(n)},H^{(N)}]=0, \tag{14}$$

where

$$F^{(n)}=f_1^{(n)}+k_1^{(n)}f_2^{(n)}+\cdots+k_1^{(n)}\cdots k_{L-1}^{(n)}f_L^{(n)}, \tag{15}$$
$$E^{(n)}=e_1^{(n)}(k_2^{(n)})^{-1}\cdots(k_L^{(n)})^{-1}+\cdots+e_{L-1}^{(n)}(k_L^{(n)})^{-1}+e_L^{(n)}, \tag{16}$$
$$K^{(n)}=k_1^{(n)}\cdots k_L^{(n)}, \tag{17}$$

and each $x_i^{(n)}$ $(x=f,e,k)$ acts nontrivially on site i. From the similarity transformation we have

$$[\widetilde{F}^{(n)},M^{(N)}]=[\widetilde{E}^{(n)},M^{(N)}]=[\widetilde{K}^{(n)},M^{(N)}]=0, \tag{18}$$

where $\widetilde{X}^{(n)}=S^{-1}X^{(n)}S$ $(X=F,E,K)$. By the direct calculation, one can show that the elements of $\widetilde{F}^{(n)}$ and $\widetilde{E}^{(n)}$ are given as

$$\langle j_1\cdots j_L|\widetilde{F}^{(n)}|k_1\cdots k_L\rangle=\begin{cases}Qq^{\#\{i'<i|j_{i'}=n\}} & (\exists i: j_i-1=k_i=n,\\ & \ j_\iota=k_\iota(\iota\neq i)),\\ 0 & (\text{otherwise}),\end{cases} \tag{19}$$

$$\langle k_1\cdots k_L|\widetilde{E}^{(n)}|j_1\cdots j_L\rangle=\begin{cases}Qq^{\#\{i'>i|k_{i'}=n+1\}} & (\exists i: j_i-1=k_i=n,\\ & \ j_\iota=k_\iota(\iota\neq i)),\\ 0 & (\text{otherwise}),\end{cases} \tag{20}$$

$$Q=q^{\frac{1}{4}(\#\{i'|j_{i'}=n\}+\#\{i'|j_{i'}=n+1\}-1)}. \tag{21}$$

We notice that the matrix $\widetilde{F}^{(n)}$ (resp. $\widetilde{E}^{(n)}$) sends a ket (resp. bra) vector in the sector $m=(m_1,\ldots,m_{N+1})$ to the sector $m^{(n)}=(m_1,\ldots,m_{n-1},m_n-1,m_{n+1}+1,m_{n+2},\ldots,m_{N+1})$, i.e. $\widetilde{F}^{(n)}V_m\subseteq$

$V_{m^{(n)}}$, $V_m^*\widetilde{E}^{(n)} \subseteq V_{m^{(n)}}^*$. In other words, these matrices change one particle n to $n+1$. From the commutation relations (18), we have

$$\widetilde{F}^{(n)} M_m = M_{m^{(n)}} \widetilde{F}^{(n)}, \quad M_m \widetilde{E}^{(n)} = \widetilde{E}^{(n)} M_{m^{(n)}}, \tag{22}$$

which implies $\mathrm{Spec}(M_m) \subseteq \mathrm{Spec}(M_{m^{(n)}})$ or $\mathrm{Spec}(M_m) \supseteq \mathrm{Spec}(M_{m^{(n)}})$ according to $\dim V_m \leq \dim V_{m^{(n)}}$ or $\dim V_m \geq \dim V_{m^{(n)}}$, respectively.[1] Here $\mathrm{Spec}(M_m)$ is the multiset of all eigenvalues of sector m, where the multiplicity of each element corresponds to the degree of degeneracy.

For a sector $m = (m_1, \ldots, m_{N+1})$, we define

$$m^{(n)^\mu} = (m_1, \ldots, m_{n-1}, m_n - \mu, m_{n+1} + \mu, m_{n+2}, \ldots, m_{N+1}), \tag{23}$$

i.e. the sector $m^{(n)^\mu}$ is obtained by changing μ particles of nth class to $(n+1)$st-class particles. The matrix $(F^{(n)})^\mu$ sends a vector in V_m to $V_{m^{(n)^\mu}}$, and one can show by induction that each element of $(\widetilde{F}^{(n)})^\mu$ is calculated as

$$\langle j_1 \cdots j_L | (\widetilde{F}^{(n)})^\mu | k_1 \cdots k_L \rangle = [\mu]!\, Q^\mu\, q^{\sum_{i \in I} \#\{i' < i | j_{i'} = n\}}, \tag{24}$$

$$\text{if } \exists I \subset \{1, \ldots, L\}\ (\#I = \mu) \text{ such that } \ j_\iota - 1 = k_\iota = n\ (\iota \in I) \text{ and } j_\iota = k_\iota (\iota \notin I), \tag{25}$$

or 0 otherwise. Here $[\mu]!$ is the q-factorial $\prod_{1 \leq \mu' \leq \mu} [\mu']$ with the q-integer $[\mu'] = 1 + q + \cdots + q^{\mu' - 1}$, and Q is defined by (21). Similarly, $(\widetilde{E}^{(n)})^\mu$ sends a vector in V_m^* to $V_{m^{(n)^\mu}}^*$, and each element is calculated as

$$\langle k_1 \cdots k_L | (\widetilde{E}^{(n)})^\mu | j_1 \cdots j_L \rangle = [\mu]!\, Q^\mu\, q^{\sum_{i \in I} \#\{i' > i | k_{i'} = n+1\}} \tag{26}$$

if the condition (25) is satisfied, or 0 otherwise. From equation (22) we have

$$(\widetilde{F}^{(n)})^\mu M_m = M_{m^{(n)^\mu}} (\widetilde{F}^{(n)})^\mu, \quad M_m (\widetilde{E}^{(n)})^\mu = (\widetilde{E}^{(n)})^\mu M_{m^{(n)^\mu}}. \tag{27}$$

We call this type of relations *conjugation relation* and the matrix that satisfies it *conjugation matrix* [4].

Comment: Since $H^{(n)}$ (10) is a symmetric matrix, we have $M^{\mathrm{T}} = S^2 M S^{-2}$. This means our process satisfies the detailed-balance condition, and the stationary-state probability is expressed as $P_0(k_1 \cdots k_L) = \langle k_1 \cdots k_L | S^{-2} | k_1 \cdots k_L \rangle = \frac{1}{Z} q^{-\frac{1}{2} \sum_{1 \leq i < j \leq L} \mathrm{sign}(k_i - k_j)}$. This stationary state can be rewritten in terms of a matrix product form, which we will achieve in another way, i.e. by using the generalized matrix ansatz, in section 5.

3 Matrix product interpretation

In this section, we write the elements of $(\widetilde{F}^{(n)})^\mu$ and $(\widetilde{E}^{(n)})^\mu$ in a matrix product form. We first define matrix-valued matrices $b^{(n)}$ and $\bar{b}^{(n)}$ of size $(N+1) \times (N+1)$ as

$$b^{(n)} = \sum_{\substack{1 \leq x \leq N+1 \\ x \neq n}} \mathbb{1} |x\rangle\langle x| + D|n\rangle\langle n| + A|n+1\rangle\langle n|, \tag{28}$$

$$\bar{b}^{(n)} = \sum_{\substack{1 \leq x \leq N+1 \\ x \neq n+1}} \mathbb{1} |x\rangle\langle x| + D|n\rangle\langle n+1| + A|n+1\rangle\langle n+1|, \tag{29}$$

where

$$D = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix}, \quad A = \begin{pmatrix} 1 & & & \\ & q & & \\ & & q^2 & \\ & & & \ddots \end{pmatrix}, \tag{30}$$

[1] For the periodic-boundary case, this spectral inclusion is not generally satisfied [6].

satisfying the relations $DA = qAD$. We also use vectors $\langle\langle w|$ and $|v\rangle\rangle$ defined as

$$\langle\langle w| = (1\ 0\ 0\ \cdots), \quad |v\rangle\rangle = \begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix}, \tag{31}$$

which satisfy $\langle\langle w|A = \langle\langle w|$ and $D|v\rangle\rangle = |v\rangle\rangle$.

The elements of $(\widetilde{F}^{(n)})^\mu$ (24) and $(\widetilde{E}^{(n)})^\mu$ (26) can be interpreted as follows: for configurations $j_1 \cdots j_L$ and $k_1 \cdots k_L$ in the sectors m and $m^{(n)^\mu}$ (23), respectively, we have

$$\langle j_1 \cdots j_L|(F^{(n)})^\mu|k_1 \cdots k_L\rangle = [\mu]!\, Q^\mu \,\langle\langle w|b^{(n)}_{j_1 k_1} \cdots b^{(n)}_{j_L k_L}|v\rangle\rangle, \tag{32}$$

$$\langle k_1 \cdots k_L|(E^{(n)})^\mu|j_1 \cdots j_L\rangle = [\mu]!\, Q^\mu \,\langle\langle w|\bar{b}^{(n)}_{k_1 j_1} \cdots \bar{b}^{(n)}_{k_L j_L}|v\rangle\rangle, \tag{33}$$

where we write $b^{(n)}_{jk} = \langle j|b^{(n)}|k\rangle$ and $\bar{b}^{(n)}_{kj} = \langle k|\bar{b}^{(n)}|j\rangle$. Equation (32) is understood as follows. Let $j_1 \cdots j_L$ and $k_1 \cdots k_L$ be configurations such that

$$j_i = k_i \quad \text{or} \quad j_i - 1 = k_i = n, \tag{34}$$

and set

$$k_{i_1} = \cdots = k_{i_{\mu+\nu}} = n,\ \mu = \#I = \#\{i|j_i - 1 = k_i = n\},\ \nu = \#\{i|j_i = k_i = n\}. \tag{35}$$

Since $b_{j_i k_i} = \mathbb{1}$ for $k_i \neq n$, we have $\langle\langle w|b^{(n)}_{j_1 k_1} \cdots b^{(n)}_{j_L k_L}|v\rangle\rangle = \langle\langle w|b^{(n)}_{j_{i_1} n} \cdots b^{(n)}_{j_{i_{\mu+\nu}} n}|v\rangle\rangle$ where $j_{i_\ell} = n, n+1$ and $b^{(n)}_{j_{i_\ell} n} = A, D$. Noting $\sum_{i \in I} \#\{i' < i|j_{i'} = n\}$ corresponds to how many times we need to exchange $DA \to AD$ for reordering the matrix product $b^{(n)}_{j_{i_1} n} \cdots b^{(n)}_{j_{i_{\mu+\nu}} n}$ to $\underbrace{A\ \cdots\ A}_{\mu}\underbrace{D\ \cdots\ D}_{\nu}$, we find

$$\langle\langle w|b^{(n)}_{j_{i_1} n} \cdots b^{(n)}_{j_{i_{\mu+\nu}} n}|v\rangle\rangle = q^{\sum_{i\in I}\#\{i'<i|j_{i'}=n\}} \langle\langle w|\underbrace{A\ \cdots\ A}_{\mu}\underbrace{D\ \cdots\ D}_{\nu}|v\rangle\rangle = q^{\sum_{i\in I}\#\{i'<i|j_{i'}=n\}}. \tag{36}$$

If the configurations do not satisfy the condition (34), we have $\langle\langle w|b^{(n)}_{j_1 k_1} \cdots b^{(n)}_{j_L k_L}|v\rangle\rangle = 0$. Noting this and equations (24) and (36) we find the matrix product interpretation (32). One can show equation (33) in the same way. Defining the matrices

$$\langle j_1 \cdots j_L|\psi_{m^{(n)^\mu},m}|k_1 \cdots k_L\rangle = \langle\langle w|b^{(n)}_{j_1 k_1} \cdots b^{(n)}_{j_L k_L}|v\rangle\rangle, \tag{37}$$

$$\langle k_1 \cdots k_L|\varphi_{m,m^{(n)^\mu}}|j_1 \cdots j_L\rangle = \langle\langle w|\bar{b}^{(n)}_{k_1 j_1} \cdots \bar{b}^{(n)}_{k_L j_L}|v\rangle\rangle, \tag{38}$$

we rewrite the conjugation relation (27) as

$$\psi_{m^{(n)^\mu},m} M_m = M_{m^{(n)^\mu}} \psi_{m^{(n)^\mu},m}, \quad M_m \varphi_{m,m^{(n)^\mu}} = \varphi_{m,m^{(n)^\mu}} M_{m^{(n)^\mu}}. \tag{39}$$

In what follows, we show these relations in another way starting from local relations different from (13).

Using the relation $DA = qAD$, one can check that the tensor products $b^{(n)} \otimes b^{(n)}$ and $\bar{b}^{(n)} \otimes \bar{b}^{(n)}$ commute with the local Markov matrix:

$$\left[b^{(n)} \otimes b^{(n)}, M^{(N)}_{\mathrm{Loc}}\right] = \left[\bar{b}^{(n)} \otimes \bar{b}^{(n)}, M^{(N)}_{\mathrm{Loc}}\right] = 0. \tag{40}$$

This leads to a global commutation relation

$$\left[\left(b^{(n)}\right)^{\otimes L}, M^{(N)}\right] = \left[\left(\bar{b}^{(n)}\right)^{\otimes L}, M^{(N)}\right] = 0. \tag{41}$$

Bookending each element between vectors $\langle\langle w|$ and $|v\rangle\rangle$ (31), we obtain (scalar-valued) matrices of size $(N+1)^L \times (N+1)^L$

$$\Psi^{(n)} = \langle\langle w| \left(b^{(n)}\right)^{\otimes L} |v\rangle\rangle, \quad \Phi^{(n)} = \langle\langle w| \left(\bar{b}^{(n)}\right)^{\otimes L} |v\rangle\rangle \tag{42}$$

which satisfy

$$\left[\Psi^{(n)}, M^{(N)}\right] = \left[\Phi^{(n)}, M^{(N)}\right] = 0. \tag{43}$$

Since $\psi_{m,m^{(n)\mu}}$ (37) and $\varphi_{m^{(n)\mu},m}$ (38) are submatrices of $\Psi^{(n)}$ and $\Phi^{(n)}$, respectively, the commutation relations (43) lead to the conjugation relations (39).

4 Relation between dynamics of different values of N

In this section, we construct a matrix which intertwines dynamics of an N-species sector $m = (m_1, \cdots, m_{N+1})$ and an $(N-1)$-species sector $m' = (m_1, \ldots, m_{n-1}, m_n + m_{n+1}, m_{n+2}, \cdots, m_{N+1})$. The dynamics of the sector m' is essentially same as that of the sector $\overline{m} = (m_1, \ldots, m_{n-1}, m_n + m_{n+1}, 0, m_{n+2}, \cdots, m_{N+1})$ by regarding particles of xth class ($x \geq n+2$) as that of $(x-1)$st class. (Note that $\overline{m}^{(n)^{m_{n+1}}} = m$.) Thus we have already known that there exit matrices $\psi_{mm'}$ and $\varphi_{m'm}$ that satisfy conjugation relations $M_m \psi_{mm'} = \psi_{mm'} M_{m'}$ and $\varphi_{m'm} M_m = M_{m'} \varphi_{m'm}$.

Now we restrict our consideration to sectors $(m_1, \ldots, m_{N+1})$ such that $m_i > 0$ for all i (*basic sector*), and introduce alternative labeling for basic sectors [6]:

$$m = (m_1, \ldots, m_{N+1}) \;\leftrightarrow\; \mathfrak{s} = \{s_1, \ldots, s_N\} \quad (1 \leq s_1 < \cdots < s_N \leq L-1) \tag{44}$$

with the correspondence $s_i = m_1 + \cdots + m_i$, or equivalently $m_i = s_i - s_{i-1}$ $(s_0 = 0, s_{N+1} = L)$. In particular, for the zero-species sector, we use the labeling

$$(L) \;\leftrightarrow\; \emptyset. \tag{45}$$

According to this correspondence, the sector m' is labeled by $\mathfrak{s} \setminus \{s_n\}$. We write $V_{\mathfrak{s}} = V_m$ and $M_{\mathfrak{s}} = M_m$ for $\mathfrak{s} \leftrightarrow m$.

First we construct $\psi_{mm'} = \psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}$ that satisfies the conjugation relation by starting with the following matrix-valued matrix $a^{(N,n)}$ of size $(N+1) \times N$, a "degenerated version" of the matrix $b^{(n)}$ (28):

$$a^{(N,n)} = \sum_{\substack{1 \leq x \leq N+1 \\ x \neq n, n+1}} \mathbb{1} |x\rangle\langle\chi_n(x)| + D|n\rangle\langle n| + A|n+1\rangle\langle n|, \tag{46}$$

where

$$\chi_n(x) = \begin{cases} x & (x \leq n), \\ x-1 & (x > n). \end{cases} \tag{47}$$

By using the relation $DA = qAD$, one can check that this matrix satisfies the following relation, a "degenerated version" of the commutation relation (40):

$$M^{(N)}_{\mathrm{Loc}} (a^{(N,n)} \otimes a^{(N,n)}) - (a^{(N,n)} \otimes a^{(N,n)}) M^{(N-1)}_{\mathrm{Loc}} = 0. \tag{48}$$

From this relation, the L-fold tensor product of $a^{(N,n)}$ satisfies

$$\sum_{1 \leq i \leq L-1} \left(M^{(N)}_{\mathrm{Loc}}\right)_{i,i+1} \left(a^{(N,n)}\right)^{\otimes L} - \left(a^{(N,n)}\right)^{\otimes L} \sum_{1 \leq i \leq L-1} \left(M^{(N-1)}_{\mathrm{Loc}}\right)_{i,i+1} = 0. \tag{49}$$

Noting that the summations of local Markov matrices are total Markov matrices, we get

$$M^{(N)}\Psi^{(N,n)} = \Psi^{(N,n)}M^{(N-1)}, \tag{50}$$

where $\Psi^{(N,n)} = \langle\!\langle w| \left(a^{(N,n)}\right)^{\otimes L} |v\rangle\!\rangle$. Thus we find the submatrix $\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}$ of $\Psi^{(N,n)}$, i.e.

$$\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}} : V_{\mathfrak{s}\setminus\{s_n\}} \to V_{\mathfrak{s}}, \quad \langle j_1 \cdots j_L|\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}|k_1 \cdots k_L\rangle = \langle\!\langle w|a^{(N,n)}_{j_1k_1} \cdots a^{(N,n)}_{j_Lk_L}|v\rangle\!\rangle, \tag{51}$$

with $a^{(N,n)}_{jk} = \langle j|a^{(N,n)}|k\rangle$, satisfies the conjugation relation

$$M_{\mathfrak{s}}\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}} = \psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}M_{\mathfrak{s}\setminus\{s_n\}}. \tag{52}$$

Each element (51) of the conjugation matrix $\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}$ becomes 0 or a power of q with its exponent corresponding to how many times we need to exchange $DA \to AD$ for reordering the matrix product to $\underbrace{A\ \cdots\ A}_{s_{n+1}-s_n}\underbrace{D\ \cdots\ D}_{s_n-s_{n-1}}$:

$$\langle j_1 \cdots j_L|\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}|k_1 \cdots k_L\rangle = \begin{cases} \exp\left(\ln q \displaystyle\sum_{i:j_i=n+1} \#\{i' < i|j_{i'} = n\}\right) & (\chi_n(j_i) = k_i\ (\forall i)), \\ 0 & (\text{otherwise}). \end{cases} \tag{53}$$

For example, for the sectors $\mathfrak{s} = \{1, 3, 5\} \leftrightarrow (1, 2, 2, 1)$ and $\mathfrak{s} \setminus \{s_2\} = \{1, 5\} \leftrightarrow (1, 4, 1)$ with $L = 6$,

$$\begin{aligned}&\langle 123423|\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_2\}}|122322\rangle = \langle\!\langle w|\mathbb{1}DA\mathbb{1}DA|v\rangle\!\rangle = \langle\!\langle w|DADA|v\rangle\!\rangle \\ &= q\langle\!\langle w|DAAD|v\rangle\!\rangle = q^2\langle\!\langle w|ADAD|v\rangle\!\rangle = q^3\langle\!\langle w|AADD|v\rangle\!\rangle = q^3.\end{aligned} \tag{54}$$

The generalized matrix (product) ansatz, i.e. the procedure (48)-(52), was introduced in [4] for the periodic boundary condition. There the right-hand side of (48) is replaced as

$$M^{(N)}_{\mathrm{Loc}}(a^{(N,n)} \otimes a^{(N,n)}) - (a^{(N,n)} \otimes a^{(N,n)})M^{(N-1)}_{\mathrm{Loc}} = a^{(N,n)} \otimes \widehat{a}^{(N,n)} - \widehat{a}^{(N,n)} \otimes a^{(N,n)} \tag{55}$$

with an auxiliary matrix $\widehat{a}^{(N,n)}$, and the conjugation matrix is constructed by taking the trace

$$\langle j_1 \cdots j_L|\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}|k_1 \cdots k_L\rangle = \mathrm{Tr}\left[a^{(N,n)}_{j_1k_1} \cdots a^{(N,n)}_{j_Lk_L}\right]. \tag{56}$$

(We set $\widehat{a}^{(N,n)} = 0$ in our case.) The families of representations for the *hat relation* (55) found in [4, 5] are more complicated than our case. For example, for $(N, n) = (3, 1)$, our representation (46) is

$$a^{(3,1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} D & 0 & 0 \\ A & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \end{array} \quad (\text{reflective}). \tag{57}$$

On the other hand, the representation found in [4, 5] is

$$a^{(3,1)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \mathbb{1}\otimes\mathbb{1} & \delta\otimes\mathbb{1} & \mathbb{1}\otimes\delta \\ A\otimes A & 0 & 0 \\ \epsilon\otimes A & \mathbb{1}\otimes A & 0 \\ \mathbb{1}\otimes\epsilon & \delta\otimes\epsilon & \mathbb{1}\otimes\mathbb{1} \end{pmatrix} \end{array} \quad (\text{periodic}), \tag{58}$$

where

$$\delta = \begin{pmatrix} 0 & c_1 & & \\ & 0 & c_2 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix}, \ \epsilon = \begin{pmatrix} 0 & & & \\ c_1 & 0 & & \\ & c_2 & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \ c_\nu = \sqrt{1 - q^\nu}, \tag{59}$$

and $\widehat{a}^{(N,n)}$ needs to be chosen as $\mathrm{diag}(1,q,q,q)a^{(3,1)}$. Note that the number of tensor products of each element in the solution for the periodic case increases as N increases. On the other hand, the solution (46) does not contain a tensor product.

The hat relation (55) is independent from boundary conditions. However, whether a representation for the algebra defined by the hat relation is practical (i.e. whether a representation allows us to construct a nontrivial conjugation matrix) depends on boundary conditions. For example, the matrix $\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}$ defined by (56) with the representation (57) is 0.

In the same way, we can construct the restricted version for φ (38), starting with the matrix $\bar{a}^{(N,n)} = \sum_{\substack{1\le x\le N+1\\ x\neq n,n+1}} \mathbb{1}|\chi_n(x)\rangle\langle x| + D|n\rangle\langle n+1|$. The matrix $\varphi_{\mathfrak{s}\setminus\{s_n\}\mathfrak{s}}$ defined by $\langle k_1\cdots k_L|\varphi_{\mathfrak{s}\setminus\{s_n\}\mathfrak{s}}|j_1\cdots j_L\rangle = \langle\langle w|\bar{a}^{(N,n)}_{k_1 j_1}\cdots\bar{a}^{(N,n)}_{k_L j_L}|v\rangle\rangle$ with $\bar{a}^{(N,n)}_{kj} = \langle k|\bar{a}^{(N,n)}|j\rangle$ satisfies the conjugation relation $\varphi_{\mathfrak{s}\setminus\{s_n\}\mathfrak{s}}M_{\mathfrak{s}} = M_{\mathfrak{s}\setminus\{s_n\}}\varphi_{\mathfrak{s}\setminus\{s_n\}\mathfrak{s}}$. This matrix has indeed trivial elements

$$\varphi_{\mathfrak{s}\setminus\{s_n\},\mathfrak{s}} = \sum |\chi_n(j_1)\cdots\chi_n(j_L)\rangle\langle j_1\cdots j_L|, \tag{60}$$

where the summation runs over all the configuration in the sector $\mathfrak{s}$. For ket vectors, this matrix "identifies" nth and $(n+1)$st class particles as a same class [6]. The matrix $\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}$ (51) sends any basis vector $|k_1\cdots k_L\rangle \in V_{\mathfrak{s}\setminus\{s_n\}}$ to vectors in $V_{\mathfrak{s}}$ such that each particle k_i keeps its position i, $(s_{n+1}-s_n)$ particles of nth class are changed to $(n+1)$st-class particles (the rest of $(s_n - s_{n-1})$ particles of nth-class are unchanged), and νth-class particles $(\nu > n)$ are changed to $(\nu+1)$st-class particles. Such vectors are restored to $|k_1\cdots k_L\rangle$ by the identification matrix $\varphi_{\mathfrak{s}\setminus\{s_n\},\mathfrak{s}}$:

$$\begin{aligned}\varphi_{\mathfrak{s}\setminus\{s_n\},\mathfrak{s}}\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}|k_1\cdots k_L\rangle &= \varphi_{\mathfrak{s}\setminus\{s_n\},\mathfrak{s}} \sum_{\substack{j_1\cdots j_L:\\ \chi_n(j_i)=k_i\ (\forall i)}} \langle\langle w|a_{j_1k_1}\cdots a_{j_Lk_L}|v\rangle\rangle|j_1\cdots j_L\rangle \\ &= \sum_{\substack{j_1\cdots j_L:\\ \chi_n(j_i)=k_i\ (\forall i)}} \langle\langle w|a_{j_1k_1}\cdots a_{j_Lk_L}|v\rangle\rangle|k_1\cdots k_L\rangle =: C|k_1\cdots k_L\rangle.\end{aligned} \tag{61}$$

Actually the constant C is independent of the configuration $k_1\cdots k_L$ and one can show

$$C = \sum_{\substack{U_i=D,A\\ \#\{i|U_i=D\}=s_n-s_{n-1}}} \langle\langle w|U_1\cdots U_{s_{n+1}-s_{n-1}}|v\rangle\rangle = \frac{[s_{n+1}-s_{n-1}]!}{[s_n-s_{n-1}]![s_{n+1}-s_n]!}. \tag{62}$$

Thus we have

$$\varphi_{\mathfrak{s}\setminus\{s_n\},\mathfrak{s}}\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}} = \frac{[s_{n+1}-s_{n-1}]!}{[s_n-s_{n-1}]![s_{n+1}-s_n]!}\mathrm{Id}_{\mathfrak{s}\setminus\{s_n\}}, \tag{63}$$

where Id is the identity matrix. The injectivity of $\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}$ follows from this relation, and we have the inclusion relation

$$\mathrm{Spec}M_{\mathfrak{s}} \supset \mathrm{Spec}M_{\mathfrak{s}\setminus\{s_n\}}. \tag{64}$$

Now we turn to the construction of the conjugation matrix between Markov matrices of N- and N'-species sectors $(N-N'=u>0)$. Let us consider the local relation, which we also call hat relation,

$$M^{(N)}_{\mathrm{Loc}}(\mathcal{X}\otimes\mathcal{X}) - (\mathcal{X}\otimes\mathcal{X})M^{(N')}_{\mathrm{Loc}} = 0. \tag{65}$$

We have already known a family of solutions to this relation:

$$\mathcal{X} = a^{(N,n_1)}\star\cdots\star a^{(N'+1,n_u)} \quad (1\le n_\ell\le N-\ell+1), \tag{66}$$

where the symbol $\star$ denotes the product $Q\star R = \left\{\sum_j Q_{ij}\otimes R_{jk}\right\}_{ik}$ for matrix-valued matrices $Q=\{Q_{ij}\}_{ij}$ and $R=\{R_{ij}\}_{ij}$. For example,

$$\mathcal{X} = a^{(3,2)}\star a^{(2,1)} = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & D & 0 \\ 0 & A & 0 \\ 0 & 0 & \mathbb{1}\end{pmatrix} \star \begin{pmatrix} D & 0 \\ A & 0 \\ 0 & \mathbb{1}\end{pmatrix} = \begin{pmatrix} \mathbb{1}\otimes D & 0 \\ D\otimes A & 0 \\ A\otimes A & 0 \\ 0 & \mathbb{1}\otimes\mathbb{1}\end{pmatrix} \tag{67}$$

is a solution to $M^{(3)}_{\mathrm{Loc}}(\mathcal{X}\otimes\mathcal{X}) = (\mathcal{X}\otimes\mathcal{X})M^{(1)}_{\mathrm{Loc}}$.

Bookending each element of $\mathcal{X}^{\otimes L}$ between $\langle\!\langle w|^{\otimes u}$ and $|v\rangle\!\rangle^{\otimes u}$, we obtain

$$\Psi = \langle\!\langle w|^{\otimes u}\mathcal{X}^{\otimes L}|v\rangle\!\rangle^{\otimes u} = \Psi^{(N,n_1)}\cdots\Psi^{(N'+1,n_u)}, \tag{68}$$

which satisfies

$$M^{(N)}\Psi = \Psi M^{(N')}. \tag{69}$$

The matrix Ψ sends a vector of an N'-species sector to an N-species sector, via $(N'+1)$-species $\to$ $(N'+2)$-species $\to\ \cdots\ \to (N-1)$-species. Each index n_ℓ specifies which class of particles splits in sending an $(N-\ell)$-species vector to an $(N-\ell+1)$-species sector.

For the N-species sector $\mathfrak{s} = \{s_1 < \cdots < s_N\}$ and the N'-species sector $\mathfrak{t} = \mathfrak{s}\setminus\{s_{\nu_1},\dots,s_{\nu_u}\}$ $(u = N - N')$, we have the conjugation matrix

$$\psi_{\mathfrak{s}\mathfrak{t}} = \psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_{\nu_1}\}}\psi_{\mathfrak{s}\setminus\{s_{\nu_1}\},\mathfrak{s}\setminus\{s_{\nu_1},s_{\nu_2}\}}\cdots\psi_{\mathfrak{t}\cup\{s_{\nu_u}\},\mathfrak{t}} \tag{70}$$

satisfying

$$M_{\mathfrak{s}}\psi_{\mathfrak{s}\mathfrak{t}} = \psi_{\mathfrak{s}\mathfrak{t}}M_{\mathfrak{t}}\,. \tag{71}$$

This is a submatrix of Ψ (68) with the choice $n_\ell = \nu_\ell - \#\{z|z<\ell,\nu_z<\nu_\ell\}$, but indeed independent of the choice. It is enough to show the simplest case $u = 2$:

$$\text{“commutativity”:}\quad \psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_\mu\}}\psi_{\mathfrak{s}\setminus\{s_\mu\},\mathfrak{s}\setminus\{s_\mu,s_\nu\}} = \psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_\nu\}}\psi_{\mathfrak{s}\setminus\{s_\nu\},\mathfrak{s}\setminus\{s_\mu,s_\nu\}}. \tag{72}$$

We suppose $\mu<\nu$, and use the explicit expression (53). The choices of n_ℓ s for the left- and right-hand sides of (72) are $(n_1,n_2) = (\mu,\nu-1)$ and (ν,μ), respectively. By the expression (53), both sides are calculated as

$$\langle j_1\cdots j_L|\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_\mu\}}\psi_{\mathfrak{s}\setminus\{s_\mu\},\mathfrak{s}\setminus\{s_\mu,s_\nu\}}|k_1\cdots k_L\rangle = \tag{73}$$
$$\begin{cases} \exp\Big(\ln q\Big(\displaystyle\sum_{i:j_i=\mu+1}\#\{i'<i|j_{i'}=\mu\} + \sum_{i:\chi_\mu(j_i)=\nu}\#\{i'<i|\chi_\mu(j_{i'})=\nu-1\}\Big)\Big) & \begin{pmatrix}\chi_{\nu-1}(\chi_\mu(j_i))\\ = k_i\ (\forall i)\end{pmatrix}, \\ 0 & \text{(otherwise)}, \end{cases}$$

$$\langle j_1\cdots j_L|\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_\nu\}}\psi_{\mathfrak{s}\setminus\{s_\nu\},\mathfrak{s}\setminus\{s_\mu,s_\nu\}}|k_1\cdots k_L\rangle = \tag{74}$$
$$\begin{cases} \exp\Big(\ln q\Big(\displaystyle\sum_{i:j_i=\nu+1}\#\{i'<i|j_{i'}=\nu\} + \sum_{i:\chi_\nu(j_i)=\mu+1}\#\{i'<i|\chi_\nu(j_{i'})=\mu\}\Big)\Big) & \begin{pmatrix}\chi_{\mu}(\chi_\nu(j_i))\\ = k_i\ (\forall i)\end{pmatrix}, \\ 0 & \text{(otherwise)}, \end{cases}$$

which are equal.

5 Stationary state

The zero-species sector $\emptyset$ consists only of the configuration $1\cdots1$. Since $M^{(0)}_{\mathrm{Loc}} = 0$, the hat relation (65) with $N' = 0$ becomes $M^{(N)}_{\mathrm{Loc}}(\mathcal{X}\otimes\mathcal{X}) = 0$. A solution to this hat relation is given as $\mathcal{X} = a^{(N,n_1)}\star a^{(N-1,n_2)}\star\cdots\star a^{(1,1)}$, and thus we obtain a stationary state in the matrix product form [1, 7, 11]: the probability $P(j_1\cdots j_L)$ of finding a configuration $j_1\cdots j_L$ can be expressed as

$$P(j_1\cdots j_L) = \frac{1}{Z}\langle\!\langle w|^{\otimes N}X_{j_1}\cdots X_{j_L}|v\rangle\!\rangle^{\otimes N}, \tag{75}$$

where

$$X_i = \langle i|a^{(N,n_1)}\star a^{(N-1,n_2)}\star\cdots\star a^{(1,1)}|1\rangle \tag{76}$$

satisfying $X_\alpha X_\beta = qX_\beta X_\alpha$ ($\alpha < \beta$) [2]. This implies that the system satisfies the detailed-balance condition as we commented in section 2.

The stationary states in the periodic boundary condition can also be written in the matrix product form [4, 19]. However the representation for the matrices (X_i's) in our case is much simpler than that of the periodic-boundary case. For example, for $N = 3$ with the choice $n_2 = n_3 = 1$, we have

$$a^{(3,1)} \star a^{(2,1)} \star a^{(1,1)} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} D\otimes D\otimes D \\ A\otimes D\otimes D \\ \mathbb{1}\otimes A\otimes D \\ \mathbb{1}\otimes \mathbb{1}\otimes A \end{pmatrix} \text{ (reflective)}, \tag{77}$$

$$a^{(3,1)} \star a^{(2,1)} \star a^{(1,1)} = \begin{pmatrix} \mathbb{1}\otimes\mathbb{1}\otimes(\mathbb{1}+\delta)+\delta\otimes\mathbb{1}\otimes A+\mathbb{1}\otimes\delta\otimes(\epsilon+\mathbb{1}) \\ A\otimes A\otimes(\mathbb{1}+\delta) \\ \epsilon\otimes A\otimes(\mathbb{1}+\delta)+\mathbb{1}\otimes A\otimes A \\ \mathbb{1}\otimes\epsilon\otimes(\mathbb{1}+\delta)+\delta\otimes\epsilon\otimes A+\mathbb{1}\otimes\mathbb{1}\otimes(\epsilon+\mathbb{1}) \end{pmatrix} \text{ (periodic)}. \tag{78}$$

The stationary state $|P_0\rangle_\mathfrak{s}$ of each sector $\mathfrak{s} = \{s_1 < \cdots < s_N\}$ is given by a product of conjugation matrices and $|1\cdots 1\rangle$ as

$$|P_0\rangle_\mathfrak{s} = \frac{1}{Z}\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_{\nu_1}\}}\cdots\psi_{\{s_{\nu_N}\},\emptyset}|1\cdots 1\rangle \tag{79}$$

with $\{\nu_1, \ldots, \nu_N\} = \{1, \ldots, N\}$. In other words, the stationary state in sector $\mathfrak{s}$ is constructed by transferring the zero-species vector to $V_\mathfrak{s}$ via sectors $\{s_{\nu_N}\} \to \{s_{\nu_N}, s_{\nu_{N-1}}\} \to \cdots \to \mathfrak{s}\setminus\{s_{\nu_1}\}$. Note that the stationary state (79) is indeed independent of the choice of intermediate sectors (i.e. independent of the choice of ν_ℓ s.) Noting equation (63), we find Z of the sector $\mathfrak{s} = \{s_1 < \cdots < s_N\}$ in the general case is given by the q-multinomial

$$\begin{aligned} Z &= \sum_{\substack{j_1\cdots j_L \\ \text{in sector } \mathfrak{s}}} P(j_1\cdots j_L) = \sum_{\substack{j_1\cdots j_L \\ \text{in sector } \mathfrak{s}}} \langle j_1\cdots j_L|P_0\rangle_\mathfrak{s} \\ &= \langle 1\cdots 1|\varphi_{\emptyset\{s_N\}}\cdots\varphi_{\mathfrak{s}\setminus\{s_1\},\mathfrak{s}}\ \psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_1\}}\cdots\psi_{\{s_N\}\emptyset}|1\cdots 1\rangle \\ &= \frac{[L]!}{[s_1-s_0]![s_2-s_1]!\cdots[s_{N+1}-s_N]!} \quad (s_0 = 0,\ s_{N+1} = L), \end{aligned} \tag{80}$$

which is the dimension of the sector $\mathfrak{s}$ for $q = 1$.

For the TASEP case $q = 0$, a unique configuration can be realized in the stationary state since $X_\alpha X_\beta = 0$ ($\alpha < \beta$). That is, the stationary state in each sector is an absorbing state, where all the particles stay in the descending order $j_1 \cdots j_L$ ($j_i \geq j_{i+1}$ for $\forall i$). On the other hand, for the SSEP case $q = 1$, all the possible configurations are realized with an equal probability.

Now we turn to the relaxation to the stationary state, where the relaxation time τ is characterized by the largest non-zero eigenvalues E as $\tau = -\mathrm{Re}\, E^{-1}$. (The largest eigenvalue is indeed 0, which corresponds to the stationary state.) We first consider the simplest sector $\{L-1\} \leftrightarrow (L-1, 1)$, i.e. $L-1$ partiles and 1 vacancy. The spectrum of $M_{\{L-1\}}$ is given by $\mathrm{Spec}(M_{\{L-1\}}) = \{0\} \cup \{-(1+q) + 2\sqrt{q}\cos\frac{k\pi}{L}|k = 1, \ldots, L-1\}$.[3] In particular the largest non-zero eigenvalue is

$$E = -(1+q) + 2\sqrt{q}\cos\frac{\pi}{L}. \tag{81}$$

The spectrum of the general one-*species* sector $\{s_1\} \leftrightarrow (s_1, L-s_1)$ ($1 < s_1 < L$) contains that of the one-*vacancy* sector $\{L-1\}$. Furthermore we expect that the largest non-zero eigenvalue of the sector

[2] The stationary state for the multis-pecies ASEP with the general hopping rule $\alpha\beta \to \beta\alpha$ (rate $\Gamma_{\alpha\beta}$) has also the matrix product form with the algebra $X_\alpha X_\beta = \frac{\Gamma_{\beta\alpha}}{\Gamma_{\alpha\beta}} X_\beta X_\alpha$, see [1] for $N = 3$. We need a $\frac{N(N+1)}{2}$-fold tensor product in the representation for this algebra, and the decomposition structure (76) no longer exists.

[3] This can be easily derived by the Bethe ansatz, and eigenvectors (except the stationary state) are given by $|P_k\rangle_{\{L-1\}} = \sum_{1\leq i\leq L-1}\left(\lambda^i + \frac{\lambda(1-q\lambda)}{1-\lambda}(q\lambda)^{-i}\right)|1\cdots 1 \overset{i\text{th}}{2} 1\cdots 1\rangle$ with $\lambda = \frac{1}{\sqrt{q}}\mathrm{e}^{\mathrm{i}k/L}$ ($k = 1, \ldots, L-1$).

$\{s_1\}$ is equal to that of $\{L-1\}$ (81), which can be checked in small systems. This observation implies that the relaxation time behaves as [20]

$$\tau \simeq (1-\sqrt{q})^{-2} \ (q<1), \ \frac{L^2}{\pi} \ (q=1), \tag{82}$$

as $L \to \infty$. The "fist excited state" $|P_1\rangle_{\{s_N\}}$, i.e. the eigenvector corresponding to the largest non-zero eigenvalue of the sector $\{s_N\}$ can be written as $|P_1\rangle_{\{s_N\}} = \psi_{\{s_N\}\{L-1\}}|P_1\rangle_{\{L-1\}}$ where $\psi_{\{s_N\}\{L-1\}}$ is constructed as (37). Recall that the spectrum of the general multi-species sector $\mathfrak{s} = \{s_1 < \cdots < s_N\}$ also contains that of the sector $\{s_N\}$, see (64). Again we expect the largest non-zero eigenvalue of the sector $\mathfrak{s}$ is identical to (81), which can be checked for sectors with small dimensions. This implies the same behavior of the relaxation time (82) as for the one-species case. The corresponding eigenvector of the sector $\mathfrak{s}$ also has the form $|P_1\rangle_{\mathfrak{s}} = \psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_{\nu_1}\}} \cdots \psi_{\{s_{N-1},s_N\},\{s_N\}}|P_1\rangle_{\{s_N\}}$. In contrast to our case, the relaxation time of the multi-species ASEP behaves as

$$\tau \sim L^{\frac{3}{2}} \ (q<1), \ L^2 \ (q=1) \tag{83}$$

in the periodic boundary condition [6, 15]. Comparing the behaviors (82) and (83), we notice that the boundary condition plays an important role.

6 Summary

We investigated a multi-species generalization of the ASEP with reflective boundaries. We found the symmetry of the Markov matrix can be interpreted as a matrix product form, constructing conjugation matrices (37) and (38) which intertwine Markov matrices of different sectors. We showed that the conjugation relations follow from the local commutation relations (40). We also considered relations between Markov matrices of different values of N. We constructed a conjugation matrix connecting dynamics of a simpler system and a more complex system by using solutions to the hat relation (55). We saw that the stationary state can be written in a product of conjugation matrices, and the first excited state can be obtained by multiplying that of one-species sector by a product of conjugation matrices. These properties are also true in the periodic boundary condition. (However the behaviors of the relaxation time to the stationary states are different in these two boundary conditions in general.)

It is remarkable that there exist several solutions (representations) $a^{(N,n)}$ to the hat relation (55) which are suitable either the reflective boundary condition or the periodic one [4, 5], and the choice of the hat matrix $\widehat{a}^{(N,n)}$ have to be changed according to the boundary conditions. At present we do not have a systematic way to find appropriate representations as well as appropriate hat matrices. Another interesting study will be applying (or generalizing) our method to the system with injection and extraction of particles [7, 11].

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References

[1] F. C. Alcaraz, S. Dasmahapatra and V. Rittenberg, N-species stochastic models with boundaries and quadratic algebras *J. Phys. A: Math. Gen.* **31** 845 (1998)

[2] F. C. Alcaraz, M. Droz, M. Henkel and V. Rittenberg, Reaction-Diffusion Processes, Critical Dynamics and Quantum Chains,

[3] F. C. Alcaraz and V. Rittenberg, Reaction-diffusion processes as physical realizations of Hecke algebras *Phys. Lett. B* **314**, 377 (1993)

[4] C. Arita, A. Ayyer, K. Mallick and S. Prolhac, Recursive structures in the multispecies TASEP, *J. Phys. A* **44**, 335004 (2011)

[5] C. Arita, A. Ayyer, K. Mallick and S. Prolhac, in preparation

[6] C. Arita, A. Kuniba, K. Sakai and T. Sawabe, Spectrum of a multi-species asymmetric simple exclusion process on a ring, *J. Phys. A* **42**, 345002 (2009)

[7] R. A. Blythe and M. R. Evans, Nonequilibrium steady states of matrix product form: A solver's guide, *J. Phys. A* **40**, R333 (2007)

[8] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge (1995)

[9] S. R. Dahmen, Reaction-diffusion processes described by three-state quantum chains and integrability, *J. Phys. A* **28**, 905 (1995)

[10] B. Derrida, Non-equilibrium steady state: fluctuations and large deviations of the density and of the current, *J. Stat. Mech* P07023 (2007)

[11] B. Derrida, M. R. Evans, V. Hakim and V. Pasquier, An exact solution of a 1D asymmetric exclusion model using a matrix formulation *J. Phys. A* **26**, 1493 (1993)

[12] H. Hinrichsen, I. Peschel and S. Sandow, On Matrix Product Ground States for Reaction-Diffusion Models, *J. Phys. A* **29**, 2643 (1996).

[13] K. Johansson, Shape Fluctuations and Random Matrices, *Commun. Math. Phys.* **209**, 437 476 (2000)

[14] M. Kardar, G. Parisi and Y. C. Zhang, Dynamic scaling of growing interfaces, *Phys. Rev. Lett.* **56**, 889 (1986)

[15] D. Kim, Bethe ansatz solution for crossover scaling function of the asymmetric XXZ chain and the Kardar-Paris-Zhang-type growth model, *Phys. Rev. E* **52**, 3512 (1995)

[16] T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, Springer, New York (1999)

[17] C. T. MacDonald, J. H. Gibss and A. C. Pipkin, Kinetics of Biopolymerization on Nucleic Acid Templates *Biopolymers* **6**, 1 (1968)

[18] J. H. H. Perk and C. L. Schultz, New families of commuting transfer matrices in q-state vertex models, *Phys. Lett.* **84A**, 407 (1981).

[19] S. Prolhac, M. R. Evans and K. Mallick, The matrix product solution of the multispecies partially asymmetric exclusion process, *J. Phys. A* **42**, 165004, (2009)

[20] G. Sandow and G. M. Schütz, On $U_q[SU(2)]$-Symmetric Driven Diffusion, *Europhysics. Lett.* **26**, 7, (1994)

[21] T. Sasamoto and H. Spohn, Exact height distributions for the KPZ equation with narrow wedge initial condition, *Nucl. Phys. B* **834**, 523 (2010)

[22] A. Schadschneider, D. Chowdhury and K. Nishinari, *Stochastic Transport in Complex Systems: From Molecules to Vehicles*, Elsevier Science, Amsterdam (2010)

[23] C. L. Schultz, Eigenvectors of the Multi-Component Generalization of the six-vertex model, *Physica* **122A**, 71 (1983).

[24] G. M. Schütz, Duality relation for asymmetric exclusion processes *J. Stat. Phys.* **86**, 1265, (1997)

[25] G. M. Schütz, Exactly solvable models for many-body systems far from equilibrium, *Phase transitions and critical phenomena*, Vol. 19, C. Domb and J. Lebowitz eds., Academic (2001)